Evaluating $\sum n^{-2}$

Charles Martin

March 29, 2010

Let's see some proofs of the mysterious fact $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. For all of these, keep the following in mind:

$$\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right) = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$$
$$\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right) = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

So if we can figure out the sum $\sum 1/n^2$ for just even n or just odd n, then we can get the whole sum.

Euler's trick. Consider $\sin x = x - x^3/6 + \cdots$ as an infintely long polynomial. It has roots at 0, $\pm\pi, \pm 2\pi, \ldots$, so what if we could factor it?

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots.$$

It turns out this infinite product converges and equals $\sin x$ for every x. Now imagine expanding the product out; what is the coefficient of x^3?

$$-\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} - \cdots$$

but on the other hand, the coefficient of x^3 should be $-1/6$ (from the Taylor series). Therefore

$$-\frac{1}{6} = -\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} - \cdots,$$

giving $\sum 1/n^2 = \pi^2/6$. $\square$

Fourier. From the Fourier series

$$x^2 = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{4} + \frac{\cos 3x}{9} - \frac{\cos 4x}{16} + \cdots\right), \qquad -\pi < x < \pi$$

Let $x = 0$ to get

$$\begin{aligned} 0 &= \frac{\pi^2}{3} - 4\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots\right) \\ &= \frac{\pi^2}{3} - 4\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right) + 8\left(\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \cdots\right) \\ &= \frac{\pi^2}{3} - 2\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right). \end{aligned}$$

We get $\sum 1/n^2 = \pi^2/6$. $\square$

Parseval. Take the 2π-periodic function

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

and expand into a Fourier series:

$$f(x) \sim \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$$

Parseval's idenity gives

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{16}{\pi^2}\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right),$$

so that

$$1 + \frac{1}{9} + \frac{1}{25} + \cdots = \frac{\pi^2}{8}.$$

As mentioned above, this gives $\sum 1/n^2 = \pi^2/6$. □

Integrals. From the series expansion

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\,\frac{x^{2n+1}}{2n+1}, \qquad -1 < x < 1$$

we get

$$x = \sin x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\,\frac{\sin^{2n+1} x}{2n+1}, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Integrate on $[0, \pi/2]$ to get

$$\frac{\pi^2}{8} = 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)\cdot(2n+1)} \int_0^{\pi/2} \sin^{2n+1} x\,dx.$$

Recalling the formula

$$\int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n+1)},$$

our series becomes

$$\frac{\pi^2}{8} = 1 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = 1 + \frac{1}{9} + \frac{1}{25} + \cdots,$$

and again we get $\sum 1/n^2 = \pi^2/6$. □

Complex analysis. Given a positive integer N, let C_N be the positively oriented square contour with vertices at $(\pm(N+1/2), \pm(N+1/2))$. However, instead of $\mathbb{R}^2$ we're working on the *complex plane.* Standard facts of complex analysis tell us that many complex line integrals can be evaluated with a sum. For our purposes we have

$$\oint_{C_N} \frac{\pi \cot \pi z}{z^2}\,dz = \lim_{z\to 0}\left(\frac{1}{z^2} - \pi^2 \csc^2 \pi z\right) + 2\sum_{n=1}^{N} \frac{1}{n^2}.$$

Furthermore you can show that

$$\lim_{N\to\infty} \oint_{C_N} \frac{\pi \cot \pi z}{z^2}\,dz = 0.$$

Altogether, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2}\lim_{z\to 0}\left(\frac{1}{z^2} - \pi^2 \csc^2 \pi z\right) = \frac{\pi^2}{6}.$$ □

An 'elementary' proof. The inequality $\sin a < a < \tan a$ is true for all small positive numbers a. Rearranging it gives $\cot a < 1/a < \csc a$. Given a positive integer N,

$$\sum_{n=1}^{N} \cot^2 \frac{n\pi}{2N+1} < \sum_{n=1}^{N} \left(\frac{2N+1}{n\pi}\right)^2 < \sum_{n=1}^{N} \csc^2 \frac{n\pi}{2N+1}.$$

Tricky trigonometry can prove the formulas

$$\sum_{n=1}^{N} \cot^2 \frac{n\pi}{2N+1} = \frac{N(2N-1)}{3}$$
$$\sum_{n=1}^{N} \csc^2 \frac{n\pi}{2N+1} = \frac{2N(N+1)}{3},$$

so our big inequality gives us

$$\frac{\pi^2 N(2N-1)}{3(2N+1)^2} < \sum_{n=1}^{N} \frac{1}{n^2} < \frac{2\pi^2 N(N+1)}{3(2N+1)^2}.$$

Taking $N \to \infty$ gives

$$\frac{\pi^2}{6} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq \frac{\pi^2}{6}. \qquad \square$$